A GAMBLING THEOREM AND OPTIMAL STOPPING THEORY

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1. Introduction.

A gambling theorem, stated by Dubins and Savage as Theorem 3.9.5 in [3], can be specialized to give results in the theory of optimal stopping. A proof of the theorem is given below in the finitely additive setting of [3]. Some results on measurability are then obtained under assumptions of countable additivity. Finally connections are made with the stopping theory of Snell [7], Chow and Robbins as in [1], Haggstrom [4], and Siegmund [6]. A generalization of the usual stopping problem is made to a situation where one is allowed to stop only at certain times along a given path. In accordance with [3], we treat only the uniformly bounded case, but the results in section 4 are no doubt true when properly formulated for a setting like that of Snell [7].

Notation is mostly borrowed from Dubins and Savage [3].

2. A Gambling Theorem.

Let F be a set and α a gamble-valued function on F. Assume Γ is a gambling house on F such that, for every f, either $\Gamma(f) = \{\alpha(f)\}$ or $\Gamma(f) = \{\alpha(f), \delta(f)\}$, where $\delta(f)$ is the gamble assigning mass one to f. Let u be a bounded utility function on F and let U and V be the corresponding utility of Γ (p. 25, [3]) and strategic utility of Γ (p. 41, [3]) respectively.

Let $\overline{\sigma}(\cdot)$ be the stationary family of strategies associated with the map

$$(1) \quad \begin{aligned} \gamma(f) &= \delta(f) \quad \text{if} \quad u(f) = V(f) \quad \text{and} \quad \delta(f) \in \Gamma(f), \\ &= \alpha(f) \quad \text{otherwise.} \end{aligned}$$

Theorem 1 (Theorem 3.9.5 of [3]): *The strategy* $\bar{\sigma}(f)$ *is optimal for every* f. *That is,* $u(\bar{\sigma}(f)) = V(f)$ *for every* f.

The strategies $\bar{\sigma}(f)$ are clearly thrifty (Theorem 3.6.1, [3]). So it suffices to prove that they are also equalizing (Theorem 3.5.1, [3]). The following two lemmas are helpful.

Lemma 1: *Suppose* $\Gamma'(f) = \{\alpha(f), \delta(f)\}$ *for every* f *and also* $u' = 1_B$ *is an indicator function. Let* $\bar{\tau}(\cdot)$ *be the stationary family determined by*

$$\beta(f) = \alpha(f) \quad \text{if} \quad f \notin B \tag{2}$$

$$= \delta(f) \quad \text{if} \quad f \in B.$$

Then, for every f, $\bar{\tau}(f)$ *is optimal in* Γ' *at* f.

Proof: Let $Q(f) = u'(\bar{\tau}(f))$ and apply Theorem 2.12.1 of [3]. □

Lemma 2: *Let* Γ *be any gambling house on* F, u *a utility function, and* U *the corresponding utility of* Γ. *Let* $\varepsilon > 0$ *and define* $B = \{f: u(f) \geq U(f) - \varepsilon\}$. *Then, for the new gambling problem with utility function* $u' = 1_B$, Γ *has corresponding utility* U' *identically equal to* 1.

Proof: Let $0 < \varepsilon' < \varepsilon$ and let $f \in F$. If $u(f) = U(f)$, then clearly $U'(f) = 1$. So assume $u(f) < U(f)$. Choose σ available at f and a stop rule t such that $u(\sigma, t) > U(f) - (\varepsilon')^2$. Since U is excessive, $U(f) \geq U(\sigma, t)$. Hence, $(U - u)(\sigma, t) < (\varepsilon')^2$. Since $U \geq u$, we have

$$\sigma[U(f_t) - u(f_t) \geq \varepsilon] \leq \sigma[U(f_t) - u(f_t) \geq \varepsilon'] \leq \varepsilon'.$$

Thus $U'(f) \geq 1_B(\sigma, t) \geq 1 - \epsilon'$. $\square$

A somewhat deeper result is that Lemma 2 would remain true if U and U' were replaced by V and V' in its statement.

Now we return to the proof that $\overline{\sigma}(f)$ is equalizing.

Let $\epsilon > 0$ and s be any stop rule. By Theorem 3.7.2 of [3], it suffices to find a stop rule t such that $t \geq s$ and $\overline{\sigma}(f)[f_t \epsilon A] \geq 1 - \epsilon$, where $A = \{f: u(f) \geq V(f) - \epsilon\}$.

Let $g \epsilon F$, $\overline{\tau}(g)$ be as in Lemma 1, and B be as in Lemma 2. Then $A \supseteq B$, since $V \leq U$, and, by the lemmas,

$$1_A(\overline{\tau}(g)) \geq 1_B(\overline{\tau}(g)) = U'(g) = 1.$$

In particular, there is a stop rule $r(g)$ such that

$$\overline{\tau}(g)[f_{r(g)} \epsilon A] \geq 1 - \epsilon.$$

Let

$$t_\epsilon(f_1, \ldots) = \text{least } k, \text{ if any, for which } f_k \epsilon A$$
$$= +\infty \text{ if all } f_k \notin A.$$

We may assume $r(g) \leq t_\epsilon$. For, if not, we could replace $r(g)$ by $r(g) \wedge t_\epsilon$ and observe that

$$[f_{r(g)\wedge t_\epsilon} \epsilon A] \supseteq [f_{r(g)} \epsilon A].$$

But $\overline{\sigma}(g)$ agrees with $\overline{\tau}(g)$ up to time t_ϵ. Hence,

$$\overline{\sigma}(g)[f_{r(g)} \epsilon A] \geq 1 - \epsilon.$$

Now let $h = (f_1, \ldots)$ and suppose $s(h) = n$. Define

$$t(h) = n + r(f_n)(f_{n+1}, \ldots).$$

Then

$$\bar{\sigma}(f)[f_t \in A] = \int \bar{\sigma}(f_{s(h)})[f_{r(f_{s(h)})} \in A] d\bar{\sigma}(f)(h) \quad \text{(formula 3.7.1, [3])}$$
$$\geq 1 - \epsilon.$$

This completes the proof of Theorem 1.

Consider now the stationary family $\bar{\sigma}_\epsilon(\cdot)$ determined by

$$\gamma_\epsilon(f) = \delta(f) \quad \text{if} \quad u(f) \geq V(f) - \epsilon \quad \text{and} \quad \delta(f) \in \Gamma(f),$$
$$= \alpha(f) \quad \text{otherwise.}$$

Let t_ϵ be the time at which $\bar{\sigma}_\epsilon(f)$ stagnates, the same t_ϵ which occurs in the proof of Theorem 1.

<u>Theorem 2</u>: <u>For every</u> f, $u(\bar{\sigma}_\epsilon(f)) \geq V(f) - \epsilon$. <u>Moreover, if</u> $\delta(f) \in \Gamma(f)$ <u>for all</u> f, <u>then</u> $\bar{\sigma}_\epsilon(f)[t_\epsilon < +\infty] = 1$ <u>for all</u> f.

<u>Proof</u>: Clearly, $\bar{\sigma}_\epsilon(f)$ is thrifty. So $V(\bar{\sigma}_\epsilon(f)) = V(f)$. By Lemma 2, given $\epsilon' > 0$ and a stop rule s, we can find a stop rule $t \geq s$ such that $\bar{\sigma}_\epsilon(f)[u(f_t) > V(f_t) - \epsilon] \geq 1 - \epsilon'$. It follows that $u(\bar{\sigma}_\epsilon(f)) \geq V(\bar{\sigma}_\epsilon(f)) - \epsilon$.

The last part of the theorem follows easily from Lemma 2. □

<u>Theorem 3</u>: <u>For all</u> f <u>in</u> F,

$$V(f) = \max\{u(f), \int V d\alpha(f)\} \quad \text{if} \quad \delta(f) \in \Gamma(f)$$
$$= \int V d\alpha(f) \quad \text{if} \quad \delta(f) \notin \Gamma(f).$$

Proof: Easy using the fact that V is excessive for Γ (p. 41, [3]).

3. A Countably Additive Setting.

The new assumptions for this section are that

(a) a Borel field $\mathcal{B}$ of subsets of F is given;

(b) each gamble γ available in Γ is countably additive when restricted to $\mathcal{B}$ and each γ is identified with its restriction to $\mathcal{B}$;

(c) the map α, of the previous section is a regular conditional probability on $(F, \mathcal{B})$ in the sense that the map $f \to \alpha(f)(A)$ is $\mathcal{B}$-measurable for every $A \in \mathcal{B}$;

(d) $\{f: \delta(f) \in \Gamma(f)\} \in \mathcal{B}$;

(e) the utility function u is $\mathcal{B}$-measurable. Under these regularity assumptions, we have

Theorem 4: The strategic utility function V is $\mathcal{B}$-measurable and, hence, the map γ (defined in (1)) is $\mathcal{B}$-measurable.

Before the proof, a definition is necessary. Let $\sigma = \sigma_0, \sigma_1, \ldots$ be a strategy. Suppose σ_0 restricted to $\mathcal{B}$ is countably additive and, for every $n \geq 1$ and every n-tuple $(f_1, \ldots, f_n)$ of elements of F, $\sigma_n(f_1, \ldots, f_n)$ restricted to $\mathcal{B}$ is countably additive. Suppose also that, for $n \geq 1$ and $A \in \mathcal{B}$, $\sigma_n(f_1, \ldots, f_n)(A)$ is a $\mathcal{B} \times \ldots \times \mathcal{B}$(n-factors) measurable function of $(f_1, \ldots, f_n)$. Then σ is said to be a measurable strategy. Theorem 4 implies that the strategies $\bar{\sigma}(f)$ of section 2 are measurable, since $\bar{\sigma}(f)_n(f_1, \ldots, f_n) = \gamma(f_n)$. Thus, for our problem, the optimal strategy is measurable, although in other measurable problems the question of existence of good measurable strategies remains open (cf. [9]).

A measurable strategy σ determines a probability measure on the measurable sets $\mathcal{B} \times \mathcal{B} \times \ldots = \mathcal{B}^{\infty}$ of $F \times F \times \ldots = H$ as well as on the finitary sets. These measures are consistent and have a common extension (section 2 of [9]) which we also write as σ.

Define u^* on H by

$$(3) \qquad u^*(f_1, f_2, \ldots) = \limsup_{n \to \infty} u(f_n).$$

According to Theorem 3.2 of [9],

$$(4) \qquad u(\sigma) = \int u^* d\sigma \quad \text{for every measurable strategy } \sigma.$$

<u>Lemma 3</u>: <u>Let</u> σ <u>be a measurable strategy</u>. <u>Then</u>

$$u(\sigma[f_1, \ldots, f_n]) \to u^*(f_1, \ldots, f_n, \ldots) \ \sigma\text{ - a.s.}$$

<u>as</u> $n \to \infty$. (Recall that $\sigma[f_1, \ldots, f_n]$ denotes the conditional strategy determined by σ given the first n fortunes are $f_1, \ldots, f_n$.)

<u>Proof</u>: By (4),

$$u(\sigma[f_1, \ldots, f_n]) = \int u^* d\sigma[f_1, \ldots, f_n].$$

Since u^* is shift invariant, the right hand expression is just the conditional expectation of u^* under σ given $f_1, \ldots, f_n$. The lemma now follows from a version of the martingale convergence theorem (Theorem VII.4.3 of [2]). □

The next lemma is a special case of Theorem 4.

Lemma 4: *If* $\delta(f) \in \Gamma(f)$, *for all* f, *then* V *is* $\mathcal{B}$-*measurable*.

Proof: Let $U_0(f) = u(f)$ and, for $n = 1, 2, \ldots, U_{n+1}(f) = \max(U_n(f),$ $\int U_n d\alpha(f))$.

Then each U_n is $\mathcal{B}$-measurable. By Theorem 2.15.5 and Corollary 3.3.2 of [3], $U_n \to V$ as $n \to \infty$. □.

It is easy to generalize Lemma 4 to the case where there are countably many gambles available at each f as in Theorem 4.1 of [8].

Now we are ready to prove Theorem 4.

Consider a gambling house Γ' on F given by $\Gamma'(f) = \{\alpha(f), \delta(f)\}$ for every f. Let $\bar{\lambda}(\cdot)$ be the stationary family of measurable strategies determined by the map α. Define

$$u'(f) = \int u^* d\bar{\lambda}(f) \quad \text{if} \quad \delta(f) \notin \Gamma(f)$$

$$= \max\{u(f), \int u^* d\bar{\lambda}(f)\} \quad \text{if} \quad \delta(f) \in \Gamma(f).$$

Let V' be the corresponding strategic utility. Then, by Lemma 4, V' is $\mathcal{B}$-measurable. (It is straightforward to check the measurability of u'.) Thus it suffices to show $V = V'$.

For every f, $\bar{\lambda}(f)$ is available at f in Γ. By (4),

$V(f) \geq u(\bar{\lambda}(f)) = \int u^* d\bar{\lambda}(f)$. Hence, $V \geq u'$. But V is excessive for Γ and thus for Γ'. By Theorem 2.12.1, [3], $V \geq V'$.

It remains to prove that $V \leq V'$. Fix f in F and let $\gamma(f)$ and $\bar{\sigma}(f)$ be as in section 2. By Theorem 1, $u(\bar{\sigma}(f)) = V(f)$. Now $\bar{\sigma}(f)$ is available at f in Γ'. So it suffices to show

(5) $u'(\bar{\sigma}(f)) \geq u(\bar{\sigma}(f))$.

(In fact, equality holds.)

Let $C = \{f: \gamma(f) = \delta(f)\}$ and define

$$t_C(f_1,\ldots) = \text{least } k, \text{ if any, for which } f_k \in C$$
$$= +\infty \text{ if } f_k \notin C \text{ for all } k.$$

Then, for any stop rule t,

$$(6) \quad u(\bar{\sigma}(f), t) = u(\bar{\sigma}(f), t \wedge t_C)$$
$$= u(\bar{\lambda}(f), t \wedge t_C) \text{ (by Theorems 3.4.3 and 3.4.4 of [3])}$$
$$= \int_E u(f_t)d\bar{\lambda}(f) + \int_{E^c} u(f_{t_C})d\bar{\lambda}(f),$$

where $E = [t \leq t_C]$.

Let $\epsilon > 0$.

For each positive integer n, let $B_n = \{h: \exists\, k \ni k \geq n$ and $u(f_k) > u^*(h) + \epsilon\}$. Then $B_n \downarrow \phi$ and $\bar{\lambda}(f)$ is measurable, so that $\exists N_1$ with $\bar{\lambda}(f)(B_{N_1}) < \epsilon$. Thus, if $t \geq N_1$,

$$[u(f_t) > u^* + \epsilon] \subseteq B_{N_1}$$

and

$$\bar{\lambda}(f)[u(f_t) > u^* + \epsilon] < \epsilon.$$

Hence,

$$(7) \quad u(\bar{\sigma}(f), t) \leq \int_E u^* d\bar{\lambda}(f) + \int_{E^c} u(f_{t_C})d\bar{\lambda}(f) + \epsilon(2M + 1),$$

for $t \geq N_1$ and $M = \sup|u|$.

Now the equations in (6) remain true if u is replaced by u'. By Lemma 3, $u'(f_n) \geq \int u^* d\overline{\lambda}(f_n) = u(\overline{\lambda}(f_n)) \to u^*$ as $n \to \infty$ $\overline{\lambda}(f)$ - a.s. Hence, $\exists N_2 \ni \overline{\lambda}(f)\{h: \exists k \geq N_2$ and $u'(f_k) < u^*(h) - \epsilon\} \leq \epsilon$. Notice also that, for every h, $u'(f_{t_C}(h)) \geq u(f_{t_C}(h))$ since $f \in C \Rightarrow \delta(f) \in \Gamma(f)$.

Thus, for $t \geq N_2$,

$$(8) \qquad u'(\overline{\sigma}(f), t) \geq \int_E u^* d\overline{\lambda}(f) + \int_{E^C} u(f_{t_C}) d\overline{\lambda}(f) - \epsilon(2M + 1).$$

By (7) and (8),

$$u'(\overline{\sigma}(f), t) \geq u(\overline{\sigma}(f), t) - 2\epsilon(2M + 1),$$

for $t \geq \max(N_1, N_2)$, which proves (5) and, hence, the theorem.

4. Stopping Theory.

Let $(Y_n, \mathcal{B}_n)_{n\geq 1}$ be a sequence of measurable spaces and, for $n = 1, 2, \ldots$, let

$$(Y^n, \mathcal{B}^n) = (Y_1 \times \ldots \times Y_n, \mathcal{B}_1 \times \ldots \times \mathcal{B}_n)$$

and

$$(Y^\infty, \mathcal{B}^\infty) = (Y_1 \times \ldots, \mathcal{B}_1 \times \ldots).$$

Let P be a countably additive probability measure on $\mathcal{B}^\infty$. We shall assume, for every n, the existence of a regular conditional distribution of the $n + 1$st coordinate y_{n+1} given the first n coordinates $y_1, \ldots, y_n$. The assumption is not very restrictive in practice and the existence is guaranteed if the $(Y_n, \mathcal{B}_n)$ are separable standard Borel spaces (Theorem V.8.1, [5]).

For $n \geq 1$, let X_n be a $\mathcal{B}^n$-measurable map from Y^n to the Borel line and assume the X_n are uniformly bounded.

A <u>stopping variable</u> (sv) is a random variable t on $(Y^\infty, \mathcal{B}^\infty, P)$ with range contained in $\{1, 2, \ldots, +\infty\}$ and such that, for every two elements $y = (y_1, y_2, \ldots)$ and $y' = (y_1', y_2', \ldots)$ of Y^∞, if $t(y) = n$ and $y_i = y_i'$ for $i \leq n$, then $t(y') = n$. Stopping variables are not assumed here to be finite with probability one. Following [6], we define, for $y \in Y^\infty$,

$$X_t(y) = X_{t(y)}(y) \quad \text{if} \quad t(y) < \infty,$$

$$= \limsup_{n\to\infty} X_n(y) \quad \text{if} \quad t(y) = \infty.$$

The object is to choose a sv which maximizes EX_t.

We generalize the problem by restricting the sv's allowed. Let $A_n \in \mathcal{B}^n$ for $n \geq 1$. A sv t is <u>permissible</u> iff, for every $y = (y_1, \ldots) \in Y^\infty$, $t(y) = n$ implies $(y_1, \ldots, y_n) \in A_n$. (To specialize to the case where all sv's are permissible, take $A_n = Y^n$ for all n.) The object now is to find the optimal sv among the class of permissible sv's. It will be seen that an optimal sv always exists. To obtain this result, we consider an associated gambling problem.

Define

$$F = \left(\bigcup_{n=1}^{\infty} Y^n\right) \cup \{f_0\} \quad \text{where} \quad f_0 \notin \bigcup_{n=1}^{\infty} Y^n;$$

$$\mathcal{B} = \text{Borel field generated by } \left(\bigcup_{n=1}^{\infty} \mathcal{B}^n\right) \cup \{\{f_0\}\};$$

(The unions above can be assumed to be unions of disjoint sets.)

$\alpha(f_0)$ = distribution of y_1 ;

$\alpha(y_1,\ldots, y_n)$ = a version of the regular conditional distribution of $(y_1,\ldots, y_n, y_{n+1})$ given $(y_1,\ldots, y_n)$ for $n = 1, 2,\ldots;$

$\Gamma(f_0) = \{\alpha(f_0)\};$

$$\Gamma(y_1,\ldots, y_n) = \{\alpha(y_1,\ldots, y_n)\} \text{ if } (y_1,\ldots, y_n) \notin A_n,$$

$$= \{\alpha(y_1,\ldots, y_n), \delta(y_1,\ldots, y_n)\}$$

if $(y_1,\ldots, y_n) \in A_n$ for $n = 1, 2,\ldots;$

(For each $f \in F$, $\alpha(f)$ can be extended, by the Hahn-Banach Theorem, to all subsets of F so as to be a gamble. The particular extension taken is irrelevant for the sequel.)

$u(f_0)$ is an arbitrary real number;

$u(y_1,\ldots, y_n) = X_n(y_1,\ldots, y_n)$ for all $(y_1,\ldots, y_n) \in Y^n$ and all $n = 1, 2,\ldots.$

The gambling problem defined above is of the type studied in section 3.

Now associate to each sv t a gamble-valued function γ_t defined on F by

(9) $\gamma_t(f_0) = \alpha(f_0)$

$$\gamma_t(y_1,\ldots, y_n) = \delta(y_1,\ldots, y_n) \text{ if } t(y_1,\ldots, y_n,\ldots) = n$$

$$\gamma_t(y_1,\ldots, y_n) = \alpha(y_1,\ldots, y_n) \text{ if } t(y_1,\ldots, y_n,\ldots) \neq n.$$

Let σ_t be the associated stationary strategy at f_0 and notice that σ_t is a measurable strategy.

Lemma 5: For every sv t, $u(\sigma_t) = EX_t$. If t is permissible, then σ_t is available at f_0 in Γ.

Proof: Let $h = (f_1, f_2, \ldots) \in H$ and define

$$Y_t(h) = X_n(f_n) \quad \text{if} \quad f_n = (y_1, \ldots, y_n) \in Y^n \quad \text{and} \quad t(y_1, \ldots, y_n, \ldots) = n,$$

$$= \limsup_{n \to \infty} X_n(f_n) \quad \text{if} \quad f_n \in Y^n \quad \text{for all} \quad n.$$

Then Y_t is defined on a set of H which has probability one under σ_t. Moreover,

$$\int X_t dP = \int Y_t d\sigma_t,$$

since the distribution of X_t under P is the same as that of Y_t under σ_t.

By Theorem 3.2 of [9],

$$u(\sigma_t) = \int u^* d\sigma_t,$$

where u^* is as in (3). Since $\sigma_t[u^* = Y_t] = 1$, the first statement of the lemma is proved. The second is obvious. □

Consider the function s on Y^∞ given by

$$s(y_1, y_2, \ldots) = \text{least } n, \text{ if any, such that } u(y_1, \ldots, y_n) = V(y_1, \ldots, y_n)$$
$$\text{and } \delta(y_1, \ldots, y_n) \in \Gamma(y_1, \ldots, y_n),$$
$$= +\infty \quad \text{if there is no such } n.$$

Here V is the strategic utility function for Γ.

The next result is the principal one in this section and overlaps with theorems in [1], [4], [6], and [7].

Theorem 5: The function s is a permissible sv and is optimal.

Proof: By Theorem 4, V is $\mathcal{B}$-measurable. It follows that s is $\mathcal{B}^{\infty}$-measurable and, therefore, a sv. Clearly, s is permissible.

Let $\bar{\sigma}(f_0)$ be the optimal stationary strategy of Theorem 1. Then $\bar{\sigma}(f_0)$ and σ_s agree on a set of histories which has probability one under both. So, by Lemma 5,

$$EX_s = u(\sigma_s) = u(\bar{\sigma}(f_0)) = V(f_0).$$

But, for every permissible sv t,

$$EX_t = u(\sigma_t) \leq V(f_0),$$

again by Lemma 5. □

It is worth remarking that the proof shows σ_s to be optimal among a much larger class of strategies than just the class of σ_t arising from permissible sv's t.

Other results which overlap with previous work in stopping theory can now be easily established. Theorem 2 can be reinterpreted for this section to give information about ϵ-optimal sv's and Theorem 3 gives a functional equation for V. Finally, an easy application of the fundamental theorem of gambling (Theorem 2.12.1, [3]) shows that, if $A_n = Y^n$ for all n then $V(y_1), V(y_1, y_2), \ldots$ is minimal among the class of all expectation decreasing semi-martingales (measurable or not) adapted to $(Y^n, \mathcal{B}^n)$ and satisfying $V(y_1, \ldots, y_n) \geq X_n(y_1, \ldots, y_n)$ for all $(y_1, \ldots, y_n)$. Viewed from this standpoint, Snell's original work in [7], where he used "maximal semi-martingales," seems very much in the spirit of the basic gambling result.

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